Robust Stability

MEM 355 Performance Enhancement of Dynamical Systems

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Outline

- Stability & Robustness
- Introduction – role of sensitivity functions
- Nyquist Analysis
- Traditional gain/phase margins

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Introduction to Nyquist Stability Analysis

- Nyquist Analysis
  - A graphical method to determine how many closed loop poles are in the right half plane
  - Developed in the early 1930's – those days it was not easy to find the roots of high order polynomials
- Stability Margins
  - Nyquist analysis provides a clear concept of 'stability margin'
  - This concept generalizes to more complex MIMO systems
  - It remains a key concept in the current era

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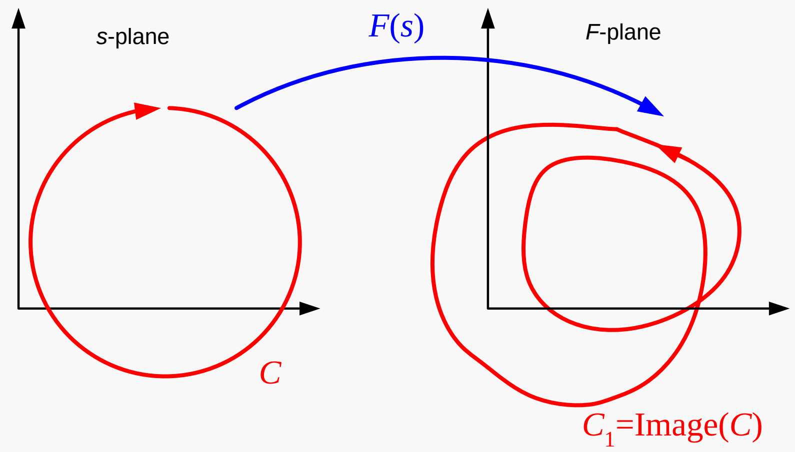
Cauchy Theorem

Any function of a complex number can be viewed as a map of points in one complex plane to another



Theorem (Cauchy): Let C be a simple closed curve in the s-plane. $F(s)$ is a rational function, having neither poles nor zeros on C. If C_1 is the image of C under the map $F(s)$, then

$$Z = P - N$$

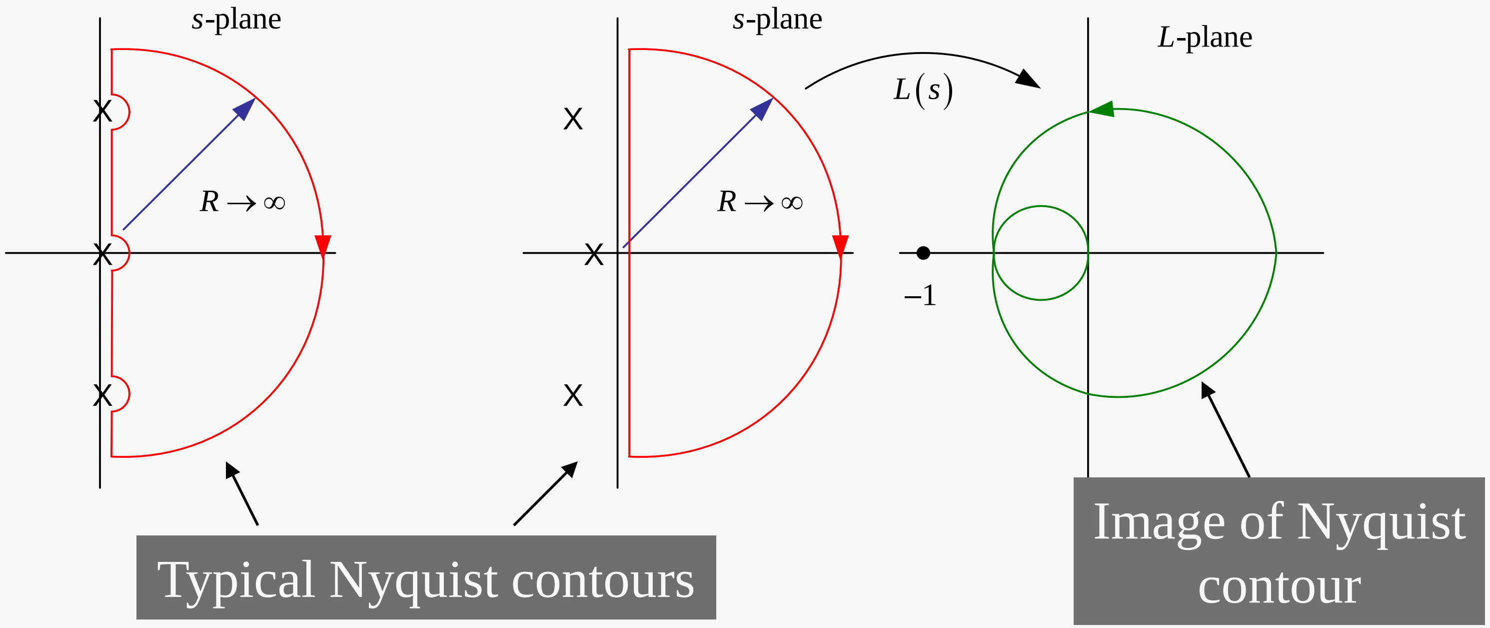
where

- N the number of counterclockwise encirclements of the origin by C_1 as s traverses C once in the clockwise direction.
- Z the number of zeros of $F(s)$ enclosed by C, counting multiplicities.
- P the number of poles of $F(s)$ enclosed by C, counting multiplicities.

Nyquist

S, Sensitivity function

- Take $F(s)=1+L(s)$ (note: $F=S^{-1}$)
- Choose a C that encloses the entire RHP
- Map into L-plane instead of F-plane (shift by -1)



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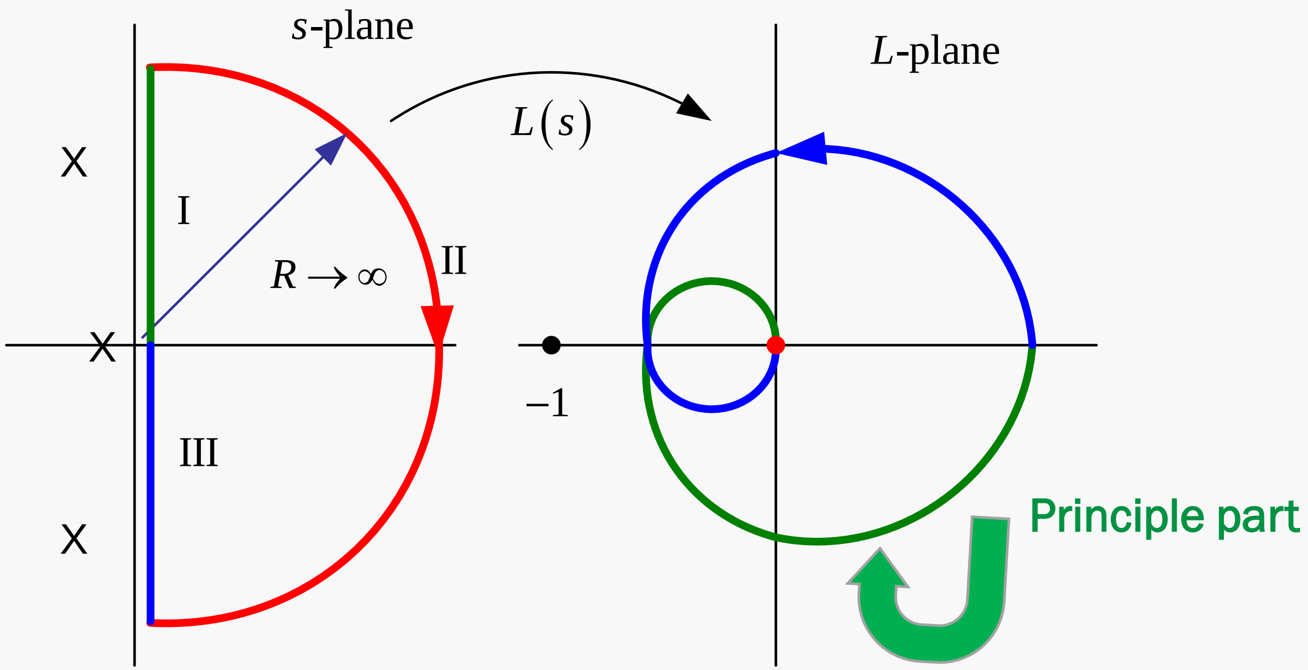
Nyquist Theorem

Theorem (Nyquist): If the plot of $L(s)$ (i.e., the image of the Nyquist contour in the L-plane) encircles the point -1+j0 in the counterclockwise direction as many times as there are unstable open loop poles (poles of $L(s)$ within the Nyquist contour) then the feedback system has no poles in the RHP.

$$Z = P - N$$

closed loop poles in RHP = open loop poles in RHP - cc encirclements of -1

Example 1



$$L(s)=\frac{1}{(s+p_1)(s^2+2\zeta\omega_0 s+\omega_0{}^2)}$$

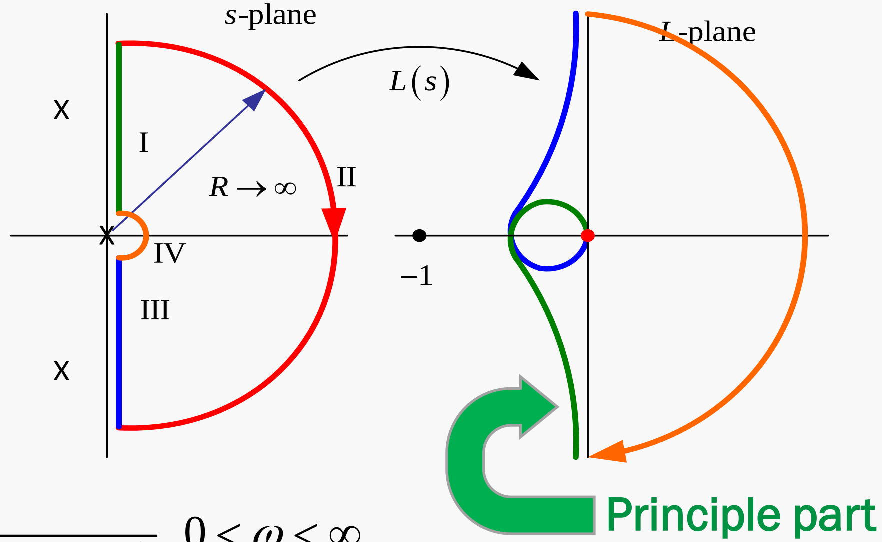
$$\text{I:}\quad s=j\omega \Rightarrow L(j\omega)=\frac{1}{(j\omega+p_1)(-\omega^2+2\zeta\omega_0 j\omega+\omega_0{}^2)},\ 0<\omega<\infty$$

$$\text{II:}\quad s=\rho\,\mathrm{e}^{j\theta},\ \rho\to\infty,\ =\frac{\pi}{2}\downarrow-\frac{\pi}{2}$$

$$L(\rho\,\mathrm{e}^{j\theta})=\frac{1}{(\rho\,\mathrm{e}^{j\theta}+p_1)(\rho^2\,\mathrm{e}^{j2\theta}+2\zeta\omega\rho\,\mathrm{e}^{j\theta}+\omega^2)}\xrightarrow[\rho\to\infty]{}0$$

$$\text{III:}\quad s=-j\omega,\quad \text{III}\to\text{I}^*$$

Example 2



$$L(s)=\frac{1}{s\left(s^2+2\zeta\omega_0 s+\omega_0{}^2\right)}$$

$$\text{I:}\quad s=j\omega \Rightarrow L(j\omega)=\frac{1}{j\omega\left(-\omega^2+j2\zeta\omega_0\omega+\omega_0{}^2\right)},\ 0<\omega<\infty$$

$$\text{II:}\quad s=\rho\,\mathrm{e}^{j\theta},\ \rho\to\infty,\ \theta=\frac{\pi}{2}\downarrow-\frac{\pi}{2}$$

$$L\left(\rho\,\mathrm{e}^{j\theta}\right)=\frac{1}{\rho\,\mathrm{e}^{j\theta}\left(\rho^2\,\mathrm{e}^{j2\theta}+2\zeta\omega_0\rho\,\mathrm{e}^{j\theta}+\omega_0{}^2\right)}\xrightarrow[\rho\to\infty]{}0$$

$$\text{III:}\quad s=-j\omega,\quad \text{III}\to\text{I}^*$$

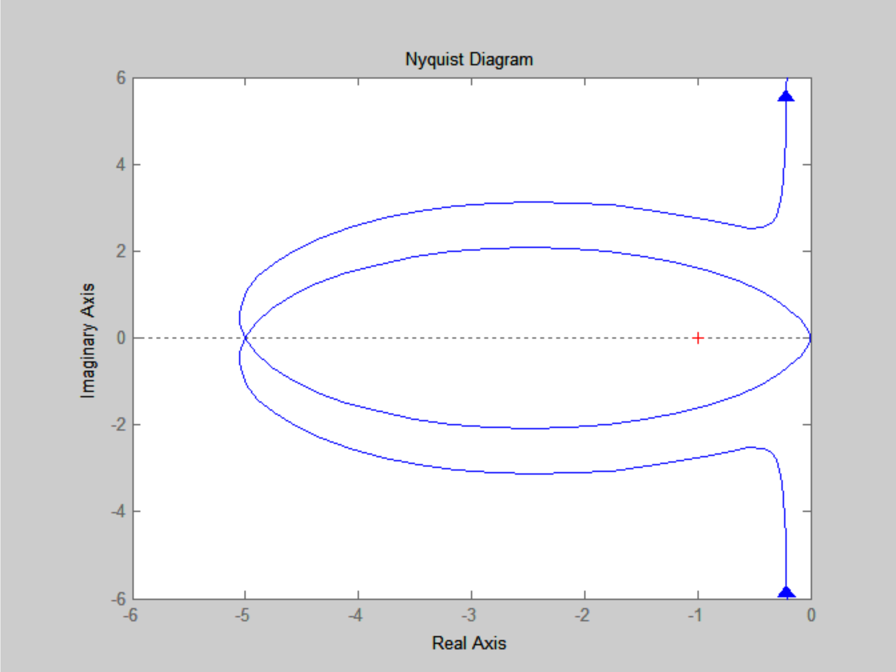
$$IV:\quad s=\varepsilon\,\mathrm{e}^{j\theta},\ \varepsilon\to 0,\ \theta=-\frac{\pi}{2}\uparrow\frac{\pi}{2}$$

$$L\left(\varepsilon\,\mathrm{e}^{j\theta}\right)=\frac{1}{\varepsilon\,\mathrm{e}^{j\theta}\left(\varepsilon^2\,\mathrm{e}^{j2\theta}+2\zeta\omega_0\varepsilon\,\mathrm{e}^{j\theta}+\omega_0{}^2\right)}\xrightarrow[\varepsilon\to 0]{}\frac{1}{\varepsilon\omega_0^2}e^{-j\theta}$$

Example 3

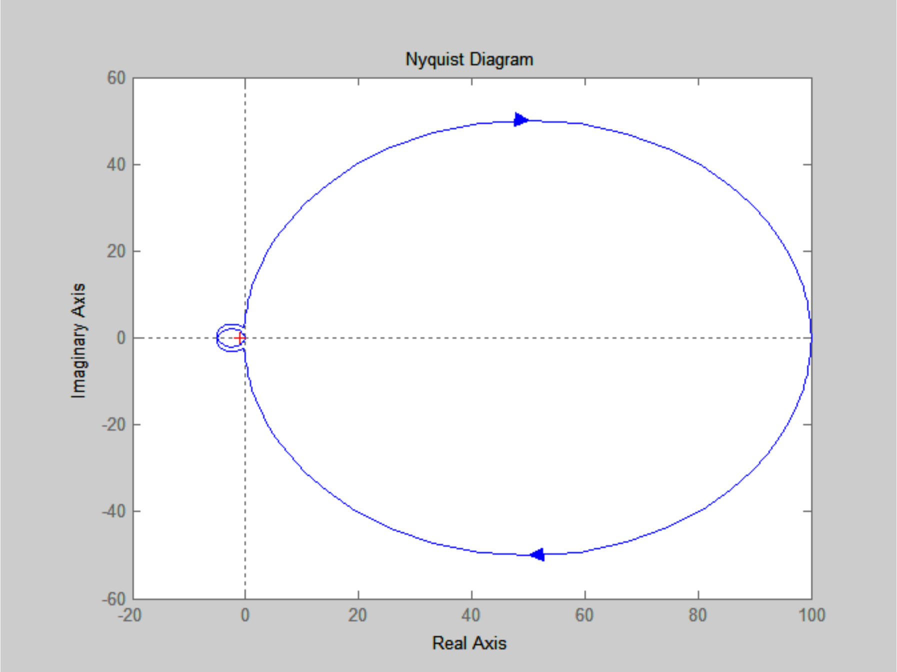
$$G(s) = \frac{1}{s\left(s^2 + 2(0.1)s + 1\right)}$$

```
>> s=tf('s');
>> G=1/(s*(s^2+2*0.1*s+1));
>> nyquist(G)
```




```
>> s=tf('s');
>> G=1/((s+0.01)*(s^2+2*0.1*s+1));
>> nyquist(G)
```

To obtain global picture



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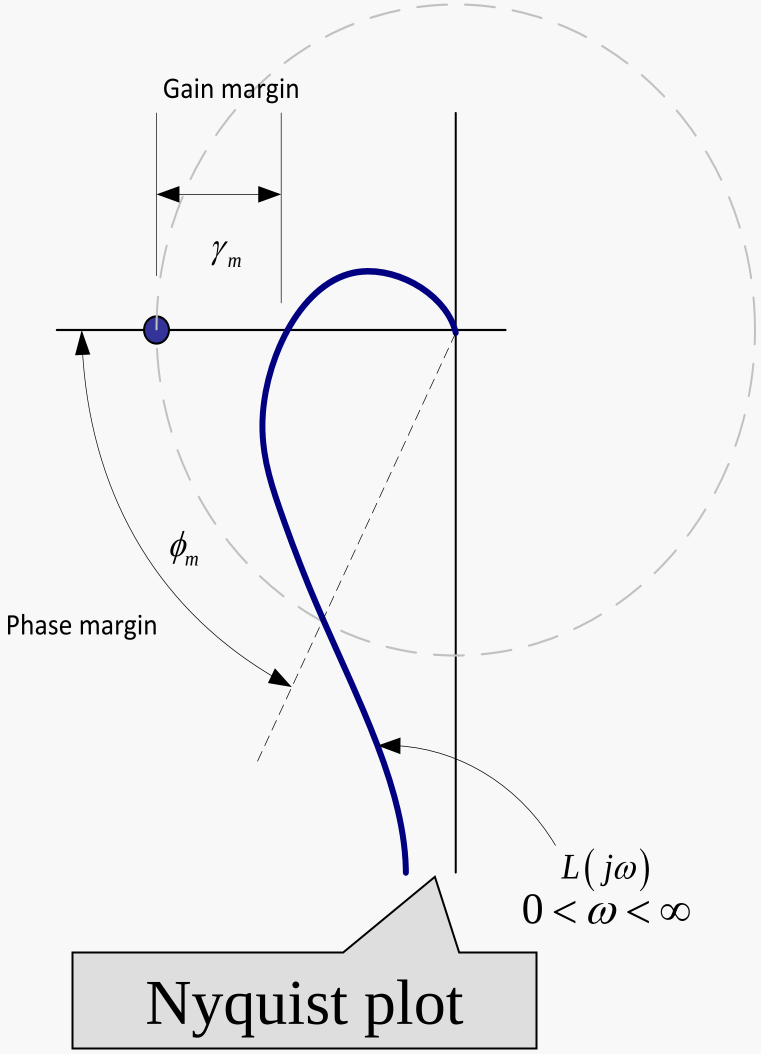
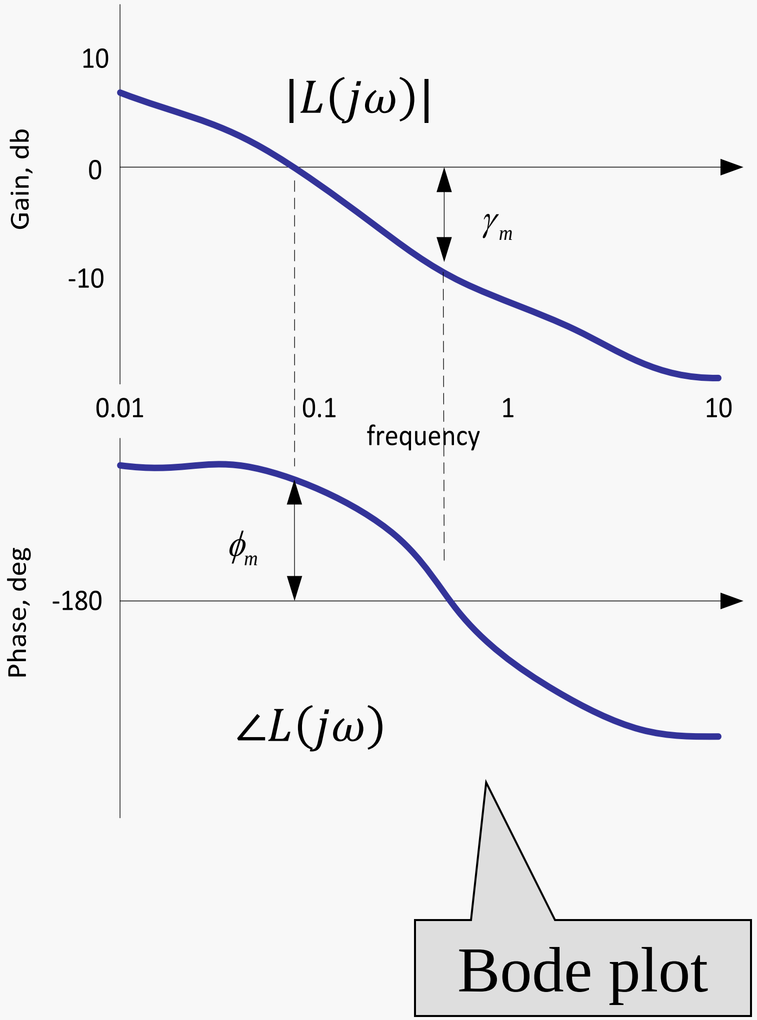
Gain & Phase Margin





Robustness From Sensitivity Functions

Sensitivity peaks are related to gain and phase margin.
Sensitivity peaks are related to overshoot and damping ratio.

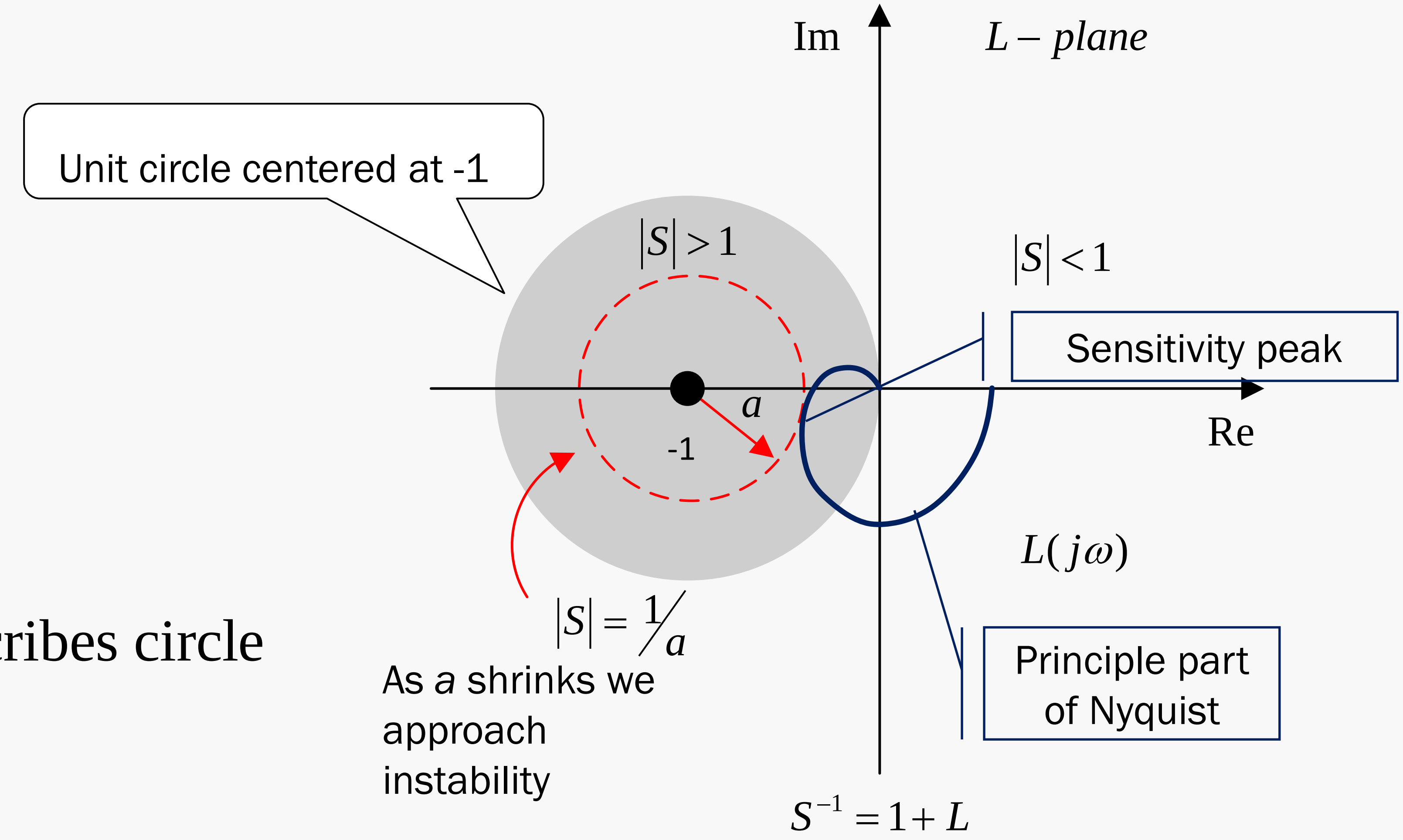
$$M_S = \max_{\omega} |S(j\omega)|$$

$$M_T = \max_{\omega} |T(j\omega)|$$

$$S = \rho e^{j\theta} = (1+L)^{-1}$$

$$\Rightarrow L = -1 + \rho^{-1} e^{-j\theta}$$

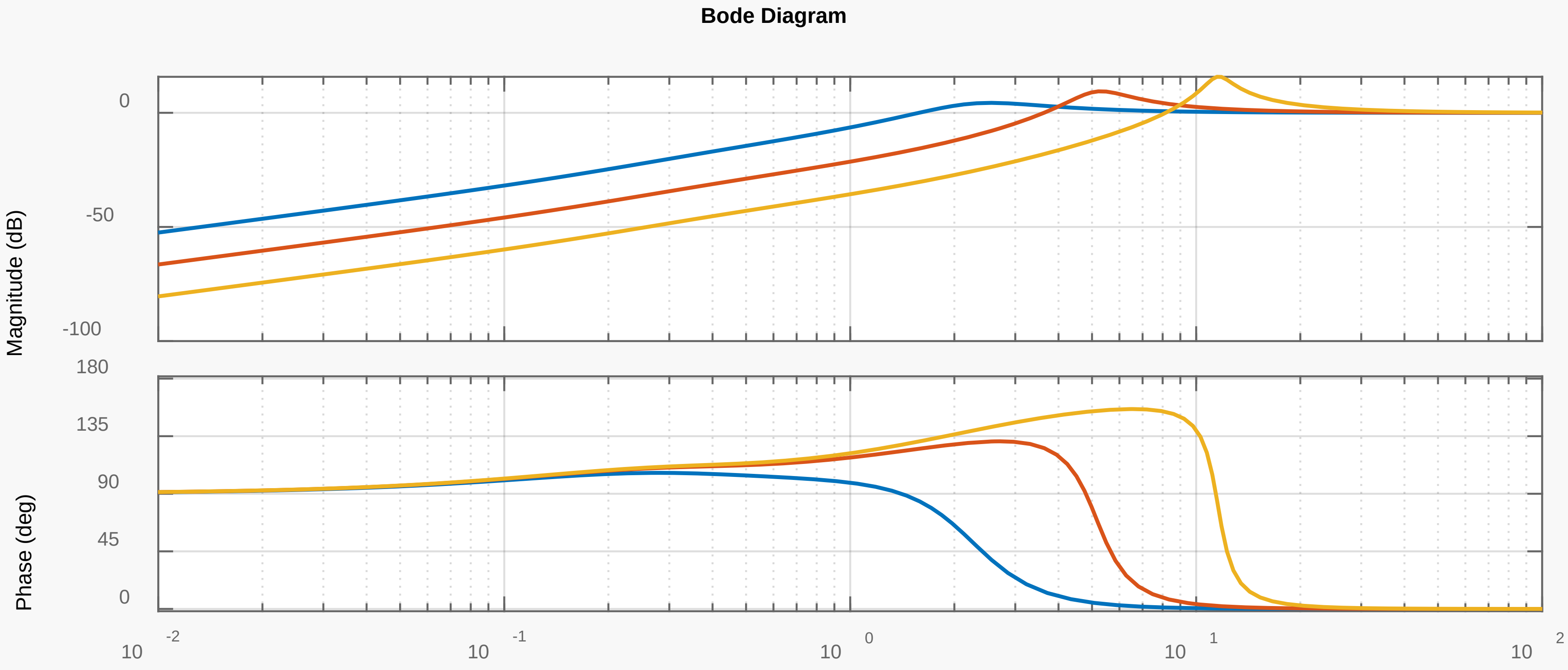
constant $|S|$ prescribes circle



Example: XV-15

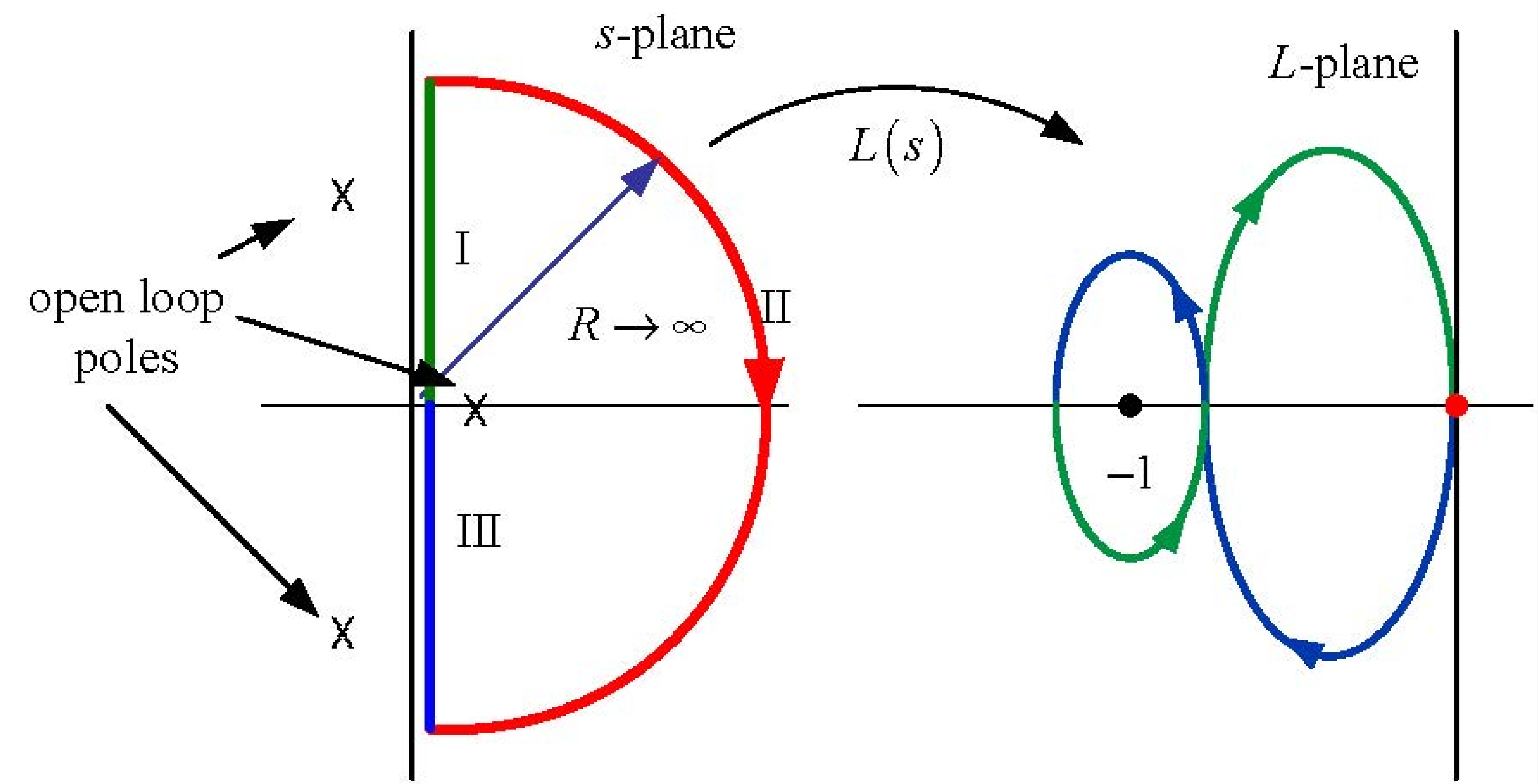
Sensitivity function plots for *K*=1, 5, 25

Larger sensitivity peak, closer to instability, reduced gain and phase margins, reduced damping, increased overshoot



Example: single unstable pole

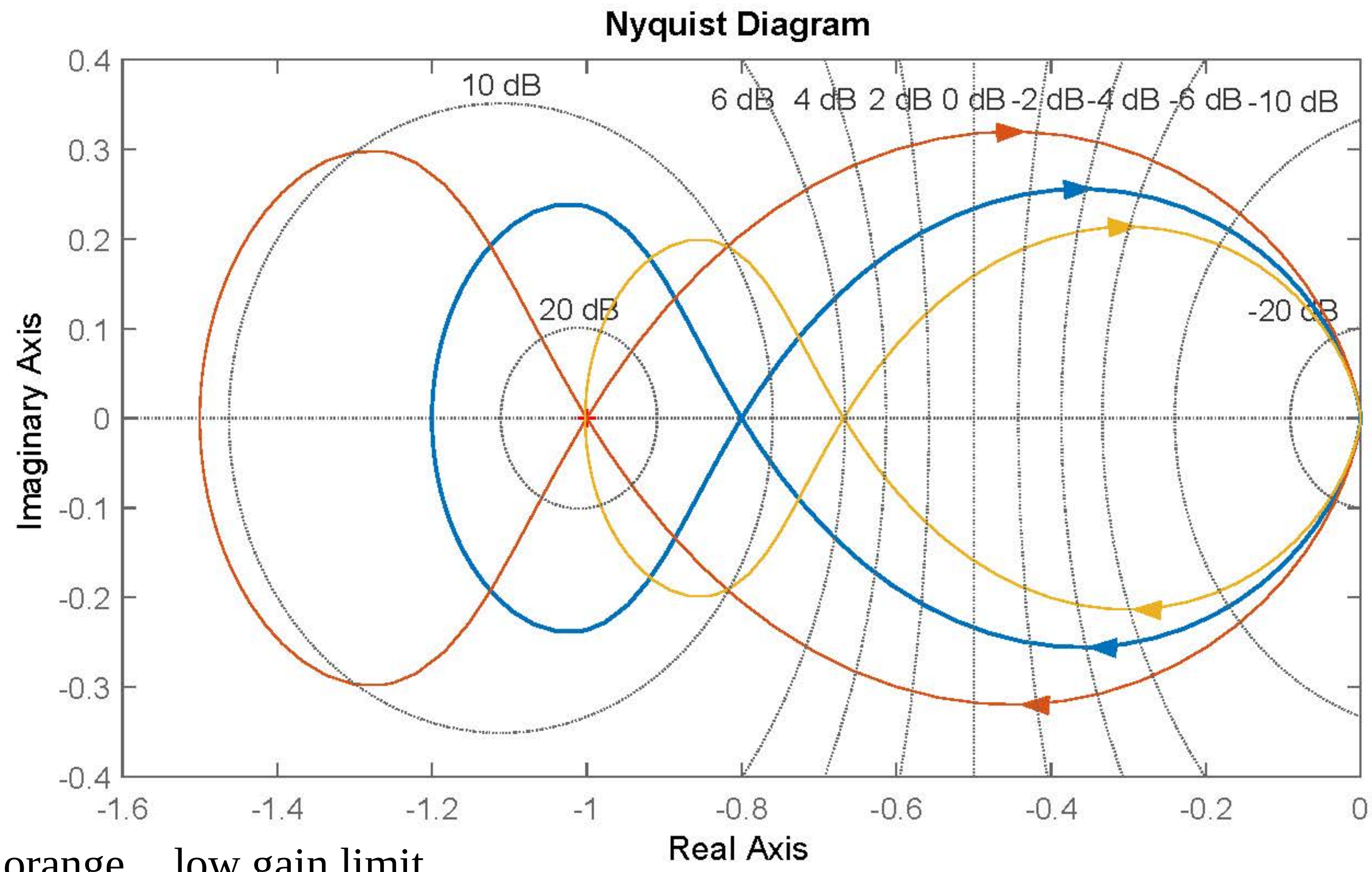
$$L(s) = \frac{1}{(s - p_1)(s^2 + 2\rho\omega_0 + \omega_0^2)}$$



Example Cont'd- MATLAB Computations

$$L(s)=\frac{0.6}{(s-0.5)(s^2+s+1)}$$

```
s=tf('s')
L=0.60/((s-0.5)*(s^2 + s + 1));
figure
nyquist(L,1.25*L,0.835*L)
grid
```



$0.835L(s)$	orange	low gain limit
$L(s)$	blue	stable
$1.25L(s)$	red	high gain limit

Summary

- Need to consider 2-3 transfer functions to fully evaluate performance
- Bandwidth is inversely related to settling time
- Sensitivity function peak is related to overshoot and inversely to damping ratio
- Gain and phase margins can be determined from Nyquist or Bode plots
- Sensitivity peak is inversely related to stability margin
- Design tools:
  - Bode and/or Nyquist diagrams helps establish robustness (margins) & performance (sensitivity peaks)